Into. To Lattice QCD

Haiyun Lu
Outline

- Introduction to QCD and path integral
- Lattice, scalar field
- Gauge field
- Lattice fermion
- Quenched approximation
- Some results
QCD

- Quantum field theory of strong interaction
- Color field group SU(3)
- Different from QED because of gluon-gluon interactions
- At high energies:
  - small coupling constant
  - perturbation theory can apply
- At low energies:
  - large coupling constant
  - perturbation theory does not apply
QCD Lagrangian

\[ L_{\text{QCD}} = - \frac{1}{4} G^\mu_\nu A^a_\mu A^a_\nu + \sum_f q_f [i \gamma^\mu D^\mu - m_f] q_f \]

with the gluon field strength tensor

\[ G^\mu_\nu = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f^{bc}_a A^\mu_b A^\nu_c \]

and the gauge covariant derivative

\[ D^\mu = \partial^\mu - i \frac{g}{2} A^\mu_a \lambda^a \]

where \( A^\mu_a \) is the gluon field, \( g \) is the strong coupling constant and \( f \) denotes the quark flavor. Looks very similar to QED, except for the last term in the second equation.
Path Integral

• Expectation value of an observable

\[ \langle O \rangle = \frac{1}{Z} \int d\psi d\overline{\psi} dA O e^{iS} \]

\[ Z = \int d\psi d\overline{\psi} dA e^{iS} \]

\[ S = \int dx L \]
Perturbation Theory

- Calculate Feynman diagrams.
- Stop at certain order.
- Order corresponds to number of vertices.
- Proportional to coupling constant, only applicable for small coupling constant.

I. Allison, “Matching the Bare and MS Charm Quark Mass using Weak Coupling Simulations”, presentation at Lattice 2008
Intrinsic QCD Scale

- Running coupling constant.
- Intrinsic QCD scale $\Lambda_{\text{QCD}}$ in the order of 1 GeV.
- Scale below which the coupling constant becomes so large that standard perturbation theory no longer applies.
- Many unresolved questions about low-energy QCD.
- This is where Lattice QCD comes in!

\[ \alpha_s(\mu) \equiv \frac{g_s^2(\mu)}{4\pi} \]

Lattice QCD

- Proposed by Wilson, 1974.
- Nonperturbative low-energy solution of QCD.
- E.O.M. discretized on 4d Euclidean space-time lattice.
- Quarks and gluons can only exist on lattice points and travel over connection lines.
- Solved by large scale numerical simulations on supercomputers.
Lattice

From continuum to discretized lattice:

\[ x_0 = -ix_4 \]
\[ S_E = iS_M \]

\[ \int d^4x \rightarrow a^4 \sum_n \]

- \( n \) four-vector that labels the lattice site, \( a \) lattice constant
- Check, take an appropriate continuum limit \((a \rightarrow 0)\) to get back the continuum theory.

Scalar Fields on a lattice

Scalar field lives on a lattice site

\[ \phi(x) \rightarrow \phi(na) \]

\[ n = (n_1, n_2, \cdots, n_D) \]

Derivative

\[ \partial_\mu \phi(x) \rightarrow \]

\[ \frac{\phi((n + \mu)a) - \phi((n - \mu)a)}{2a} \]

Dimensionless field

\[ \phi_L(n) = a\phi(na) \]
Scalar field action

- **Scalar field** $\Phi(x)$, action of continuum field theory in Euclidean space:

$$S(\Phi) = \int d^4x \left[ \frac{1}{2} \left( \partial_{\mu} \Phi \right)^2 + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4} \Phi^4 \right]$$

- **Discretize to a lattice:**

$$\int d^4x \rightarrow a^4 \sum_n$$

$$\Phi(x) \rightarrow \Phi_n$$

$$\partial_{\mu} \Phi(x) \rightarrow \frac{1}{a} \left( \Phi_{n+\mu} - \Phi_n \right)$$

- **Result:**

$$S(\Phi) = \sum_n \left\{ \frac{a^2}{2} \sum_{\mu=1}^4 \left( \Phi_{n+\mu} - \Phi_n \right)^2 + a^4 \left( \frac{m^2}{2} \Phi_n^2 + \frac{\lambda}{4} \Phi_n^4 \right) \right\}$$
Expectation value calculation

- Feynman path integral formalism
- Expectation value of an operator
  \[
  \langle 0 | O(\Phi_{n_1}, \Phi_{n_2}, ..., \Phi_{n_l}) | 0 \rangle = \frac{1}{Z} \int \prod_n [d\Phi_n] O(\Phi_{n_1}, \Phi_{n_2}, ..., \Phi_{n_l}) e^{-S(\Phi)}
  \]
  where
  \[
  Z = \int \prod_n [d\Phi_n] e^{-S(\Phi)}
  \]
- Rescale fields: \( \Phi'_n = \sqrt{\lambda} \Phi_n \)
- Lattice action becomes: \( S(\Phi) = \frac{1}{\lambda} S'(\Phi') \)
Statistical Mechanics

- **Rescaled expectation value**

\[
< 0 | O(\phi', \phi', ... \phi') | 0 > = \frac{1}{Z'} \int \prod_n [d\phi']O(\phi', \phi', ... \phi') \exp\{-\frac{1}{\lambda} S'(\phi')\}
\]

\[
Z' = \int \prod_n [d\phi'] \exp\{-\frac{1}{\lambda} S'(\phi')\}
\]

- **Recognizable?**

- **Canonical ensemble**

\[
\frac{1}{\lambda} \rightarrow \beta = \frac{1}{kT}
\]

- **Similar for fermion fields**
### Statistical Mechanics

The equivalences between a Euclidean field theory and Classical Statistical Mechanics.

<table>
<thead>
<tr>
<th>Euclidean Field Theory</th>
<th>Classical Statistical Mechanics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>Hamiltonian</td>
</tr>
<tr>
<td>unit of action $\hbar$</td>
<td>units of energy $\beta = 1/kT$</td>
</tr>
<tr>
<td>Feynman weight for amplitudes $e^{-S/\hbar} = e^{-\int \mathcal{L} dt/\hbar}$</td>
<td>Boltzmann factor $e^{-\beta H}$</td>
</tr>
<tr>
<td>Vacuum to vacuum amplitude $\int \mathcal{D}\phi e^{-S/\hbar}$</td>
<td>Partition function $\sum_{\text{conf}} e^{-\beta H}$</td>
</tr>
<tr>
<td>Vacuum energy</td>
<td>Free Energy</td>
</tr>
<tr>
<td>Vacuum expectation value $\langle 0</td>
<td>\mathcal{O}</td>
</tr>
<tr>
<td>Time ordered products</td>
<td>Ordinary products</td>
</tr>
<tr>
<td>Green’s functions $\langle 0</td>
<td>T[\mathcal{O}_1 \ldots \mathcal{O}_n]</td>
</tr>
<tr>
<td>Mass $M$</td>
<td>correlation length $\xi = 1/M$</td>
</tr>
<tr>
<td>Mass-gap</td>
<td>exponential decrease of correlation functions</td>
</tr>
<tr>
<td>Mass-less excitations</td>
<td>spin waves</td>
</tr>
<tr>
<td>Regularization: cutoff $\Lambda$</td>
<td>lattice spacing $a$</td>
</tr>
<tr>
<td>Renormalization: $\Lambda \to \infty$</td>
<td>continuum limit $a \to 0$</td>
</tr>
<tr>
<td>Changes in the vacuum</td>
<td>phase transitions</td>
</tr>
</tbody>
</table>

R. Gupta, “Introduction to Lattice QCD”, arXiv:hep-lat/9807028
Monte Carlo Method

- Method from statistical mechanics to calculate expectation value numerically.
- Generate random distribution.
- Calculate expectation value for this distribution.
- Repeat this process very many times.
- Average over results.
- Results have statistical errors.
- A lot of computational power needed!
Supercomputers

Gauge Fields on a lattice

(continuum) Gauge Fields

$$A_\mu(x) = A_\mu^a(x) T^a \in SU(N)$$

$$\text{tr } T^a = 0, \quad (T^a)^\dagger = T^a \quad \Rightarrow \quad T^a : \text{ Traceless, Hermite}$$

$$\text{tr } T^a T^b = \frac{1}{2} \delta^{ab} \quad \text{orthogonality, normalization}$$

$$[T^a, T^b] = i f^{abc} T^c, \quad f^{abc} \text{ structure constant of group } SU(N)$$

Problem: $A_\mu$ is not gauge covariant!

$$A_\mu(x) \rightarrow \frac{\Omega(x) \partial_\mu \Omega^\dagger(x)}{ig} + \Omega(x) A_\mu(x) \Omega^\dagger(x)$$

$\Omega(x) \in SU(N):$ gauge tr.
Gauge Fields on a lattice II

**Link variables**

\[ U_{n,\mu} = \exp[i g a A_\mu(n)] \in \text{SU}(N) \]

\[ U_{n+\hat{\mu},-\mu} = U_{n,\mu}^\dagger \]

**Gauge transformation**

\[ U_{n,\mu} \rightarrow U_{n,\mu}^g = g_n U_{n,\mu} g_{n+\hat{\mu}}^\dagger \quad \text{covariant} \]
Gauge Invariance III

Product of link variables

\[ \prod U \equiv U_{n,\mu_1} U_{n+\hat{\mu}_1,\mu_2} \cdots U_{m-\hat{\mu}_k,\mu_k} \longrightarrow g_n \prod U g_m^\dagger \]

Closed loop \( C \) at \( n \)

\[ \prod_C U \rightarrow g_n \left\{ \prod_C U \right\} g_n^\dagger \]

\[ \text{tr} \left\{ \prod_C U \right\} \text{ is gauge invariant} \]
Wilson Loops

Closed paths on the 4d Euclidean space-time lattice

Matrices defined on the links that connect the neighboring sites

Traces of product of such matrices along Wilson loops are gauge invariant

Plaquette: the elementary building block of the lattice, the 1 x 1 lattice square

R. Gupta, “Introduction to Lattice QCD”, arXiv:hep-lat/9807028
Wilson action

\[ S_W = \frac{1}{g^2} \text{Re} \sum_{x, \mu > \nu} \text{Tr} \frac{1}{2} (1 - U_{x,\mu} U_{x+\mu,\nu} U^{\dagger}_{x+\nu,\mu} U^{\dagger}_{x,\nu}) \]

- Simplest discretized action of the Yang-Mills part of the QCD action
- Agrees with the QCD action to order \( O(a_2) \).
- Proportional to the gauge-invariant trace of the sum over all plaquettes.
From Wilson to Yang Mills

- Matrices $U$ given by $U_\mu(x) = \exp(iagA_\mu(x + \frac{\hat{\mu}}{2}))$
- The simplest Wilson loop, the 1x1 plaquette given by

$$W_{\mu \nu}^{1x1} = U_\mu(x)U_\nu(x + \mu^\h)U_\mu^\dagger(x + \nu^\h)U_\nu^\dagger(x)$$

$$= \exp(iag[A_\mu(x + \frac{\hat{\mu}}{2}) + A_\nu(x + \mu^\h + \frac{\hat{\nu}}{2}) - A_\mu(x + \nu^\h + \frac{\hat{\nu}}{2}) - A_\nu(x + \frac{\hat{\nu}}{2})])$$

- Expanding about $x + \frac{\mu + \nu}{2}$ gives

$$= \exp[ia^2g(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{ia^4g}{12}(\partial^3_\mu A_\nu - \partial^3_\nu A_\mu) + ...]$$

- The Taylor series of the exponent gives

$$= 1 + ia^2gF_{\mu \nu} - \frac{a^4g^2}{2}F_{\mu \nu}F_{\mu \nu} + O(a^6) + ...$$

- From this we derive

$$\text{Re Tr}(1 - W_{\mu \nu}^{1x1}) = \frac{a^4g^2}{2}F_{\mu \nu}F_{\mu \nu} + ...$$
Lattice Fermions: Naive Fermion

\[ S_F = \int d^4 x \, \bar{\psi}(\gamma_\mu D_\mu + m)\psi \]

\[ S_F = a^4 \sum_n \bar{\psi}_n \left[ \sum_\mu \gamma_\mu \left( \frac{U_{n,\mu} \psi_{n+\hat{\mu}} - U_{n-\hat{\mu},\mu}^\dagger \psi_{n-\hat{\mu}}}{2a} \right) + m\psi_n \right] \]

\[ a^{3/2}\psi \rightarrow \psi, \quad ma = M \quad (\text{dimensionless}) \]

\[ S_F = \frac{1}{2} \sum_{n,\mu} \left[ \bar{\psi}_n \gamma_\mu U_{n,\mu} \psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}} U_{n,\mu}^\dagger \psi_n \right] + M \bar{\psi}_n \psi_n \]
Fermion doubling

\[ g^2 \to 0 \ (\forall U_{n,\mu} = 1) \quad S_F = \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}(-p) \left[ i\gamma_\mu \sin(p_\mu a) + M \right] \psi(p) \]

propagator

\[ G_F(p) = \frac{1}{i\gamma \cdot s + M} = \frac{-i\gamma \cdot s + M}{s^2 + M^2} \quad i\gamma \cdot s = \gamma_\mu s_\mu = \gamma_\mu \sin(p_\mu a) \]

pole of \( G_F(p) \) in \( a \to 0 \ (\hat{p}_\mu \ll 1/a) \)

\[ \sin(p_\mu a) = \begin{cases} \hat{p}_\mu a & p_\mu = \hat{p}_\mu \\ -\hat{p}_\mu a & p_\mu = \pi/a + \hat{p}_\mu \end{cases} \]

\[ \lim_{a \to 0} G_F(p) = \frac{1}{a} \sum_{p_\mu = 0, \pi/a} -i(-1)^\delta \gamma \cdot \hat{p} + m \frac{\hat{p}^2 + m^2}{\hat{p}_\mu a} \]

\[ \delta_\mu = 0 \text{ for } p_\mu = 0 \]

\[ \delta_\mu = 1 \text{ for } p_\mu = (\pi/a) \]

1 lattice fermion field \( \Rightarrow 2^d = 16 \text{ particles ("doubling problem") } \)

\[ 2^d \to \begin{cases} 2^{d-1} \quad \text{chirality } + \ (|\delta| = \text{ even}) \\ 2^{d-1} \quad \text{chirality } - \ (|\delta| = \text{ odd}) \end{cases} \]
Solution: Wilson fermions

Add $O(a)$ term $\Rightarrow$ mass to doublers ($\exists p_\mu = \pi/a$)

$$S_W = -ar \int d^4x \bar{\psi} D^2 \psi \to -\frac{r}{2} \sum_{n,\mu} [\bar{\psi}_n U_{n,\mu} \psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}} U_{n,\mu}^\dagger \psi_n - 2\bar{\psi}_n \psi_n]$$

$$S_F = S_F^0 + S_W \to \bar{\psi}(\gamma \cdot D + m)\psi \quad (a \to 0)$$

$$g^2 = 0 (U_{n,\mu} = 1)$$

$M(p)$$

$$S_F = \bar{\psi}(-p)[i\gamma \cdot s + M + r \sum_{\mu} (1 - \cos(p_\mu a))\psi(p) \Rightarrow G_F(p) = \frac{-i\gamma \cdot s + M(p)}{s^2 + M(p)^2}$$

$a \to 0$

$$M(p) = \begin{cases} 
ma & \text{for physical pole} \\
ma + 2r|\delta| & \text{for doublers}
\end{cases} \Rightarrow \begin{cases} 
m_{\text{phys}} = m \\
m_{\text{doubler}} = m + \frac{2r}{a}|\delta| \to \infty
\end{cases}$$

“decoupling of doublers at low energy”

Caution: Wilson term violates chiral symmetry
Method of operation

- Six unknown input parameters, coupling constant and the masses of the up, down, strange, charm and bottom quark.
- Top quark too short lived to form bound states at the energies we are looking at.
- Fix in terms of six precisely measured masses of hadrons.
- Masses and properties of all the other hadrons can be obtained this way.
- They should agree with experiment.
Lattice constant

- Lattice constant $a$ should be small to approach continuum limit, but not too small or the computation time becomes too long.
- Size nucleon in the order of 1 Fermi (1 Fermi = $1.0 \times 10^{-15}$ m).
- $a$ between 0.05 and 0.2 Fermi
- Results also have systematic errors due to this lattice discretization.
Quenched Approximation

- Quarks fully dynamical degrees of freedom that can be produced and annihilated.
- In the quenched approximation vacuum polarization effects of quark loops are turned off.
- Very popular approximation, reduces computation time by a factor of about $10^3$ to $10^5$.

R. Gupta, “Introduction to Lattice QCD”, arXiv:hep-lat/9807028
An Example: The Pion

• Calculate the Correlation Function
  \[ O = \left( \bar{\psi}(x) \gamma_5 \psi(x) \right) \left( \bar{\psi}(0) \gamma_5 \psi(0) \right)^+ \]

• This should behave like
  \[ C(t) = \sum_n A_n e^{-m_n t} \]

• We want to find the ground state mass
  \[ C(t) \rightarrow A_0 e^{-m_0 t} \]

As \( t \) becomes large
Mass of the ground state

- Plot $\ln\left(\frac{C(t)}{C(t+1)}\right)$ against $t$ as

$$\ln\left(\frac{C(t)}{C(t+1)}\right) = \ln\left(\frac{A_0 e^{-m_0 t}}{A_0 e^{-m_0 (t+1)}}\right) = \ln\left(e^{m_0}\right) = m_0$$

- Look for a plateau
Mass Plot

Pion Mass

Time

Pion Mass
Chiral Extrapolation
Nucleon Mass

\[ m_N(a) = m_N(0) + C_1 a \]
\[ m_N(a) = m_N + C_1 a^2 + C_2 a^2 \]

Lattice spacing
References

http://www.kvi.nl/~loehner/saf_seminar/2008/LatticeQCD.ppt


http://arxiv.org/abs/hep-lat/9807028v1

http://www.physics.gla.ac.uk/ppt/index_files/pptsymmp/PCooney.ppt

Lattice QCD (introduction) by Polikarkov DUBNA WINTER SCHOOL 1 2 FEBRUARY 2005